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# Properties of Solutions for Certain Functional Equations Arising in Dynamic Programming

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**Abstract.** In this paper, we introduce and study properties of solutions for the following functional equation arising in dynamic programming of multistage decision processes

$$f(x)=\operatorname*{opt}_{y\in D}\{u(x,y)\max\{p(x,y),f(a(x,y))\}+v(x,y)\min\{q(x,y),f(b(x,y))\}+w(x,y)[r(x,y)+f(c(x,y))]\},\quad \forall x\in S.$$

A sufficient condition which ensures the existence, uniqueness and iterative approximation of solution for the functional equation is provided. A few other behaviors of solutions for certain functional equations which are particular cases of the functional equation are discussed. The results presented in this paper extend, improve and unify the results due to Bellman, Bhakta and Choudhury, Bhakta and Mitra, Liu, and Liu and Ume. Several examples which dwell upon the importance of our results are also included.

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**Key words:** dynamic programming, functional equation, iterative approximation, nonexpansive mapping, nonnegative solution, nonpositive solution, solution

## 1. Introduction and Preliminaries

It is well known that the existence problems of solutions of various functional equations arising in dynamic programming are of both theoretical and practical interest. For details, we refer to [1–16] and the references therein. The paper of Bellman and Lee [5] provided also an excellent survey on the development and applications of the functional equations in dynamic programming.

Within the past 20 years or so, many authors, including Belbas [1], Bellman [2–4], Bellman and Roosta [6], Bhakta and Choudhury [7], Bhakta and Mitra [8], Chang [9], Chang and Ma [10], Huang et al. [11], Liu [12–14], Liu et al. [15], Liu and Ume [16] and others, investigated the existence or uniqueness of solutions, common solutions and coincidence solutions for several classes of functional equations and systems of functional equations arising in dynamic programming, by using various fixed point, common fixed point and coincidence point theorems, respectively.

In 1984, Bhakta and Mitra [8] studied the following functional equation

$$f(x) = \sup_{y \in D}\{r(x, y) + f(c(x, y))\}$$

and gave an existence and uniqueness result of solution for the functional equation. In 2001, Liu [14] obtained an existence, uniqueness and iterative approximation of nonnegative solution for the following functional equation

$$f(x) = \inf_{y \in D} \max\{p(x, y), f(a(x, y))\}.$$

In 2003, Liu and Ume [16] provided sufficient conditions which ensure the existence and uniqueness, and iterative approximation of solution for the functional equation

$$f(x) = \operatorname{opt}_{y \in D}\{u[p(x, y) + f(a(x, y))] + v \operatorname{opt}[q(x, y), f(b(x, y))]\},$$

where $u$ and $v$ are nonnegative constants with $u + v = 1$.

Motivated and inspired by the research work going on in this field, we introduce and study the following functional equation arising in dynamic programming of multistage decision processes

$$\begin{aligned} f(x) = \operatorname{opt}_{y \in D}\{&u(x, y) \max\{p(x, y), f(a(x, y))\} \\ &+ v(x, y) \min\{q(x, y), f(b(x, y))\} \\ &+ w(x, y)[r(x, y) + f(c(x, y))]\}, \quad \forall x \in S, \end{aligned} \tag{1}$$

where opt denotes sup or inf, $x$ and $y$ stand for the state and decision vectors, respectively, $a, b$ and $c$ represent the transformations of the processes, and $f(x)$ represents the optimal return function with initial $x$.

The main purpose of this paper is to discuss properties of solutions for the functional Equation (1). A sufficient condition which ensures the existence, uniqueness and iterative approximation of solution for the functional Equation (1) is provided. On the other hand, a few other behaviors of solutions for certain functional equations which are particular cases of the functional Equation (1) are established. The results presented in this paper extend, improve and unify the results due to Bellman [3], Bhakta and Choudhury [7], Bhakta and Mitra [8], Liu [14] and Liu and Ume [16]. Several examples which dwell upon the importance of our results are also included.

Throughout this paper, we assume that $R=(-\infty,+\infty)$, $R^{+}=[0,+\infty)$, $R^{-}=(-\infty,0]$ and $I$ denotes the identity mapping. Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|')$ be real Banach spaces, let $S\subseteq X$ be the state space, let $D\subseteq Y$ be the decision space. Let $BB(S)$ denote the set of all real-valued functions on $S$ that are bounded on bounded subsets of $S$. For any $k\geqslant 1$ and $f,g\in BB(S)$, let

$$d_k(f,g)=\sup\{|f(x)-g(x)|:x\in \bar{B}(0,k)\},$$
$$d(f,g)=\sum_{k=1}^{\infty}\frac{1}{2^k}\cdot\frac{d_k(f,g)}{1+d_k(f,g)},$$

where $\bar{B}(0,k)=\{x:x\in S \text{ and } \|x\|\leqslant k\}$. Then $\{d_k\}_{k\geqslant 1}$ is a countable family of pseudometrics on $BB(S)$. A sequence $\{x_n\}_{n\geqslant 1}$ in $BB(S)$ is said to be *converge* to a point $x$ in $BB(S)$ if for any $k\geqslant 1$, $d_k(x_n,x)\to 0$ as $n\to\infty$, and to be a *Cauchy sequence* if for any $k\geqslant 1$, $d_k(x_n,x_m)\to 0$ as $n,m\to\infty$. It is clear that $(BB(S),d)$ is a complete metric space. Define

$\Phi_1=\{(\varphi,\psi):\varphi$ and $\psi:R^{+}\to R^{+}$are nondecreasing, $\sum_{n=0}^{\infty}\psi(\varphi^n(t))<\infty$ and $\psi(t)>0$ for all $t>0\}$,

$\Phi_2=\{(\varphi,\psi):\varphi$ and $\psi$ are nondecreasing, $\psi(t)>0$ and $\lim_{n\to\infty}\psi(\varphi^n(t))=0$ for all $t>0\}$.

LEMMA 1.1. [16] *Let $a,b,c,d$ be in $R$. Then*

$$|\mathrm{opt}\{a,b\}-\mathrm{opt}\{c,d\}|\leqslant\max\{|a-c|,|b-d|\}.$$

## 2. Properties of Solutions for Functional Equations

In this section, we first of all study the existence, uniqueness and iterative approximation of solution for the functional equation (1). Next we investigate other properties of solutions for some functional equations which are special cases of (1). Note that the key techniques of the proof of Theorem 2.1 are to construct a nonexpansive mapping $G:BB(S)\to BB(S)$ and a Picard iteration sequence $\{z_n\}_{\geqslant 0}$ generated by (C4) such that the sequence $\{z_n\}_{\geqslant 0}$ converges to some $z$ which is both a fixed point of $G$ and a solution of the functional Equation (1). Our main results are as follows:

THEOREM 2.1. *Let $u,v,w,p,q,r:S\times D\to R$ and $a,b,c:S\times D\to S$ be mappings and let $(\varphi,\psi)$ be in $\Phi_1$ satisfying the following conditions*

$(C1)$ $|u(x,y)|+|v(x,y)|+|w(x,y)|\leqslant 1$ *for all* $(x,y)\in S\times D$;

$(C2)$ $\max\{|p(x,y)|,|q(x,y)|,|r(x,y)|\}\leqslant\psi(\|x\|)$ *for all* $(x,y)\in S\times D$;

$(C3)$ $\max\{\|a(x,y)\|,\|b(x,y)\|,\|c(x,y)\|\}\leqslant\varphi(\|x\|)$ *for all* $(x,y)\in S\times D$.

*Then the functional equation* (1) *possesses a solution $z\in BB(S)$ satisfying conditions* $(C4)$–$(C6)$

(*C*4) *the sequence* $\{z_n\}_{n\geqslant 0}$ *defined by*

$$\begin{aligned} z_0(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)p(x,y)+v(x,y)q(x,y)+w(x,y)r(x,y)\}, \quad \forall x\in S,\\ z_n(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)\max\{p(x,y), z_{n-1}(a(x,y))\}\\ &\quad +v(x,y)\min\{q(x,y), z_{n-1}(b(x,y))\}\\ &\quad +w(x,y)[r(x,y)+z_{n-1}(c(x,y))]\}, \quad \forall x\in S,\ n\geqslant 1 \end{aligned}$$

*converges to z;*

(*C*5) *if* $x_0\in S$, $\{y_n\}_{n\geqslant 1}\subset D$ *and* $x_n\in\{a(x_{n-1},y_n), b(x_{n-1},y_n), c(x_{n-1},y_n)\}$ *for all* $n\geqslant 1$, *then*

$$\lim_{n\to\infty} z(x_n)=0;$$

(*C*6) *z is unique relative to condition* (*C*5).

*Proof.* Put

$$\begin{aligned} H(x,y,h) &= u(x,y)\max\{p(x,y), h(a(x,y))\}\\ &\quad +v(x,y)\min\{q(x,y), h(b(x,y))\}\\ &\quad +w(x,y)[r(x,y)+h(c(x,y))],\\ &\quad \forall (x,y,h)\in S\times D\times BB(S), \end{aligned}$$

and

$$Gh(x)=\operatorname{opt}_{y\in D}H(x,y,h), \quad \forall (x,h)\in S\times BB(S).$$

First of all we show that

$$\varphi(t)<t, \quad \forall t>0. \tag{2}$$

Suppose that there exists some $t>0$ with $\varphi(t)\geqslant t$. Since $(\varphi,\psi)\in\Phi_1$, it follows that

$$\psi(t)\leqslant\psi(\varphi(t))\leqslant\psi(\varphi^2(t))\leqslant\cdots\leqslant\psi(\varphi^n(t))\to 0 \quad \text{as } n\to\infty.$$

That is, $\psi(t)\leqslant 0$. But $\psi(t)>0$ for $t>0$. This is a contradiction.

We claim that $G$ is a nonexpansive mapping from $BB(S)$ into itself. Let $k$ be a positive integer and let $h$ be in $BB(S)$. Using (*C*3) and (2) we see that

$$\max\{\|a(x,y)\|, \|b(x,y)\|, \|c(x,y)\|\}\leqslant\psi(\|x\|)\leqslant\|x\|\leqslant k,$$

$$\forall (x, y) \in \bar{B}(0, k) \times D,$$

which yields that there exists $g(k) > 0$ with

$$\max\{|h(a(x, y))|, |h(b(x, y))|, |h(c(x, y))|\} \leqslant g(k),$$
$$\forall (x, y) \in \bar{B}(0, k) \times D$$

and

$$\begin{aligned} |H(x, y, h)| &\leqslant |u(x, y)| \max\{|p(x, y)|, |h(a(x, y))|\} \\ &\quad + |v(x, y)| \max\{|q(x, y)|, |h(b(x, y))|\} \\ &\quad + |w(x, y)|[|r(x, y)| + |h(c(x, y))|] \\ &\leqslant (|u(x, y)| + |v(x, y)| + |w(x, y)|)(\psi(\|x\|) + g(k)) \\ &\leqslant \psi(k) + g(k), \quad \forall (x, y) \in \bar{B}(0, k) \times D \end{aligned}$$

by $(C1)$ and $(C2)$. It follows that

$$|Gh(x)| \leqslant \sup_{y \in D} |H(x, y, h)| \leqslant \psi(k) + g(k), \quad \forall x \in \bar{B}(0, k),$$

which gives that $Gh$ is bounded on bounded subsets of $S$. That is, $G$ maps $BB(S)$ into itself. Given $\epsilon > 0, x \in \bar{B}(0, k)$ and $h, t \in BB(S)$. Suppose that $\mathrm{opt}_{y \in D} = \inf_{y \in D}$. Then there exist $y, s \in D$ satisfying

$$\begin{aligned} &Gh(x) > H(x, y, h) - \epsilon, \qquad Gt(x) > H(x, s, t) - \epsilon, \\ &Gh(x) \leqslant H(x, s, h), \qquad Gt(x) \leqslant H(x, y, t). \end{aligned} \tag{3}$$

According to $(C1)$, $(C3)$, (2), (3) and Lemma 1.1, we arrive at

$$\begin{aligned} &|Gh(x) - Gt(x)| \\ &\quad < \max\{|H(x,y,h) - H(x,y,t)|, |H(x,s,h) - H(x,s,t)|\} + \epsilon \\ &\quad = \max\{|u(x,y)[\max\{p(x,y), h(a(x,y))\} - \max\{p(x,y), t(a(x,y))\}] \\ &\qquad + v(x,y)[\min\{q(x,y), h(b(x,y))\} - \min\{q(x,y), t(b(x,y))\}] \\ &\qquad + w(x,y)[h(c(x,y)) - t(c(x,y))]|,\ |u(x,s)[\max\{p(x,s), h(a(x,s))\} \\ &\qquad - \max\{p(x,s), t(a(x,s))\}] + v(x,s)[\min\{q(x,s), h(b(x,s))\} \\ &\qquad - \min\{q(x,s), t(b(x,s))\}] + w(x,s)[h(c(x,s)) - t(c(x,s))]|\} + \epsilon \\ &\quad \leqslant \max\{|u(x,y)||h(a(x,y)) - t(a(x,y))| + |v(x,y)||h(b(x,y)) - t(b(x,y))| \\ &\qquad + |w(x,y)||h(c(x,y)) - t(c(x,y))|, |u(x,s)||h(a(x,s)) - t(a(x,s))| \\ &\qquad + |v(x,s)||h(b(x,s)) - t(b(x,s))| + |w(x,s)||h(c(x,s)) - t(c(x,s))|\} + \epsilon \\ &\quad \leqslant \max\{(|u(x,y)| + |v(x,y)| + |w(x,y)|) d_k(h,t), \\ &\qquad (|u(x,s)| + |v(x,s)| + |w(x,s)|) d_k(h,t)\} + \epsilon \\ &\quad \leqslant d_k(h,t) + \epsilon, \end{aligned}$$

which means that

$$d_k(Gh, Gt) \leqslant d_k(h, t) + \epsilon.$$

Letting $\epsilon \to 0$ in the above inequality, we have

$$d_k(Gh, Gt) \leqslant d_k(h, t),$$

which implies that

$$d(Gh, Gt) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(Gh, Gt)}{1 + d_k(Gh, Gt)} \leqslant \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(h, t)}{1 + d_k(h, t)} = d(h, t). \quad (4)$$

Similarly, if $\mathrm{opt}_{y \in D} = \sup_{y \in D}$, we can conclude also that (4) holds.

We now assert that for each $x \in S$

$$|z_n(x)| \leqslant \sum_{i=0}^{n} \psi(\varphi^i(\|x\|)), \quad \forall n \geqslant 0. \quad (5)$$

In fact, $(C1)$ and $(C2)$ ensure that

$$\begin{aligned}
|z_0(x)| &= |\mathrm{opt}_{y \in D}\{u(x, y)p(x, y) + v(x, y)q(x, y) + w(x, y)r(x, y)\}| \\
&\leqslant \sup_{y \in D}\{|u(x, y)||p(x, y)| + |v(x, y)||q(x, y)| + |w(x, y)||r(x, y)|\} \\
&\leqslant \sup_{y \in D}\{(|u(x, y)| + |v(x, y)| + |w(x, y)|)\psi(\|x\|)\} \\
&\leqslant \psi(\|x\|),
\end{aligned}$$

that is, (5) holds for $n = 0$. Suppose that (5) is true for some $n \geqslant 0$. It follows from $(C1)$ to $(C3)$ that

$$\begin{aligned}
&|z_{n+1}(x)| \\
&\quad = |\mathop{\mathrm{opt}}_{y \in D}\{u(x, y)\max\{p(x, y), z_n(a(x, y))\} \\
&\qquad + v(x, y)\min\{q(x, y), z_n(b(x, y))\} + w(x, y)[r(x, y) + z_n(c(x, y))]\}| \\
&\quad \leqslant \sup_{y \in D}\{|u(x, y)|\max\{|p(x, y)|, |z_n(a(x, y))|\} \\
&\qquad + |v(x, y)|\max\{|q(x, y)|, |z_n(b(x, y))|\} \\
&\qquad + |w(x, y)|[|r(x, y)| + |z_n(c(x, y))|]\}
\end{aligned}$$

$$\leqslant \sup_{y\in D}\left\{|u(x,y)|\max\left\{\psi(\|x\|),\sum_{i=0}^{n}\psi(\varphi^i(\|a(x,y)\|))\right\}\right.$$
$$+|v(x,y)|\max\left\{\psi(\|x\|),\sum_{i=0}^{n}\psi(\varphi^i(\|b(x,y)\|))\right\}$$
$$\left.+|w(x,y)|\left[\psi(\|x\|)+\sum_{i=0}^{n}\psi(\varphi^i(\|c(x,y)\|))\right]\right\}$$
$$\leqslant \sup_{y\in D}\left\{|u(x,y)|\max\left\{\psi(\|x\|),\sum_{i=0}^{n}\psi(\varphi^{i+1}(\|x\|))\right\}\right.$$
$$+|v(x,y)|\max\left\{\psi(\|x\|),\sum_{i=0}^{n}\psi(\varphi^{i+1}(\|x\|))\right\}$$
$$\left.+|w(x,y)|\left[\psi(\|x\|)+\sum_{i=0}^{n}\psi(\varphi^{i+1}(\|x\|))\right]\right\}\leqslant\sum_{i=0}^{n+1}\psi(\varphi^i(\|x\|)).$$

Hence (5) holds for each $n\geqslant 0$.

We show that $\{z_n\}_{n\geqslant 0}$ is a Cauchy sequence in $(BB(S),d)$. Let $k$ be a positive integer and $x_0\in\bar{B}(0,k)$. Given $\epsilon>0$ and positive integers $n$ and $m$, if $\mathrm{opt}_{y\in D}=\inf_{y\in D}$, then there exist $s,t\in D$ with

$$z_n(x_0)>H(x_0,s,z_{n-1})-2^{-1}\epsilon,\qquad z_{n+m}(x_0)>H(x_0,t,z_{n+m-1})-2^{-1}\epsilon,$$
$$z_n(x_0)\leqslant H(x_0,t,z_{n-1}),\qquad z_{n+m}(x_0)\leqslant H(x_0,s,z_{n+m-1}).$$

It follows from $(C1)$ and Lemma 1.1 that there exist $y_1\in\{s,t\}$ and $x_1\in\{a(x_0,y_1),\ b(x_0,y_1),c(x_0,y_1)\}$ satisfying

$$|z_{n+m}(x_0)-z_n(x_0)|$$
$$<\max\{|H(x_0,s,z_{n+m-1})-H(x_0,s,z_{n-1})|,$$
$$|H(x_0,t,z_{n+m-1})-H(x_0,t,z_{n-1})|\}+2^{-1}\epsilon$$
$$\leqslant\max\{|u(x_0,s)||\max\{p(x_0,s),z_{n+m-1}(a(x_0,s))\}$$
$$-\max\{p(x_0,s),z_{n-1}(a(x_0,s))\}|$$
$$+|v(x_0,s)||\min\{q(x_0,s),z_{n+m-1}(b(x_0,s))\}$$
$$-\min\{q(x_0,s),z_{n-1}(b(x_0,s))\}|$$
$$+|w(x_0,s)||z_{n+m-1}(c(x_0,s))-z_{n-1}(c(x_0,s))|,$$
$$|u(x_0,t)||\max\{p(x_0,t),z_{n+m-1}(a(x_0,t))\}$$
$$-\max\{p(x_0,t),z_{n-1}(a(x_0,t))\}|$$
$$+|v(x_0,t)||\min\{q(x_0,t),z_{n+m-1}(b(x_0,t))\}$$
$$-\min\{q(x_0,t),z_{n-1}(b(x_0,t))\}|$$
$$+|w(x_0,t)||z_{n+m-1}(c(x_0,t))-z_{n-1}(c(x_0,t))|\}+2^{-1}\epsilon$$

$$
\begin{aligned}
&\leqslant \max\{|u(x_0,s)||z_{n+m-1}(a(x_0,s))-z_{n-1}(a(x_0,s))| \\
&\quad +|v(x_0,s)||z_{n+m-1}(b(x_0,s))-z_{n-1}(b(x_0,s))| \\
&\quad +|w(x_0,s)||z_{n+m-1}(c(x_0,s))-z_{n-1}(c(x_0,s))|, \\
&\quad |u(x_0,t)||z_{n+m-1}(a(x_0,t))-z_{n-1}(a(x_0,t))| \\
&\quad +|v(x_0,t)||z_{n+m-1}(b(x_0,t))-z_{n-1}(b(x_0,t))| \\
&\quad +|w(x_0,t)||z_{n+m-1}(c(x_0,t))-z_{n-1}(c(x_0,t))|\}+2^{-1}\epsilon \\
&\leqslant \max\{(|u(x_0,s)|+|v(x_0,s)|+|w(x_0,s)|) \\
&\quad \times \max\{|z_{n+m-1}(a(x_0,s))-z_{n-1}(a(x_0,s))|, \\
&\quad |z_{n+m-1}(b(x_0,s))-z_{n-1}(b(x_0,s))|, \\
&\quad |z_{n+m-1}(c(x_0,s))-z_{n-1}(c(x_0,s))|\}, \\
&\quad (|u(x_0,t)|+|v(x_0,t)|+|w(x_0,t)|) \\
&\quad \times \max\{|z_{n+m-1}(a(x_0,t))-z_{n-1}(a(x_0,t))|, \\
&\quad |z_{n+m-1}(b(x_0,t))-z_{n-1}(b(x_0,t))|, \\
&\quad |z_{n+m-1}(c(x_0,t))-z_{n-1}(c(x_0,t))|\}+2^{-1}\epsilon \\
&\leqslant |z_{n+m-1}(x_1)-z_{n-1}(x_1)|+2^{-1}\epsilon,
\end{aligned}
$$

that is,

$$
|z_{n+m}(x_0)-z_n(x_0)|<|z_{n+m-1}(x_1)-z_{n-1}(x_1)|+2^{-1}\epsilon; \tag{6}
$$

if $\text{opt}_{y\in D}=\sup_{y\in D}$, we can similarly derive that (6) holds also. Proceeding in this way, we know that there exist $y_i\in D$ and $x_i\in\{a(x_{i-1},y_i), b(x_{i-1},y_i), c(x_{i-1},y_i)\}$ for $i\in\{2,3,\dots,n\}$ such that

$$
\begin{aligned}
|z_{n+m-1}(x_1)-z_{n-1}(x_1)| &< |z_{n+m-2}(x_2)-z_{n-2}(x_2)|+2^{-2}\epsilon, \\
|z_{n+m-2}(x_2)-z_{n-2}(x_2)| &< |z_{n+m-3}(x_3)-z_{n-3}(x_3)|+2^{-3}\epsilon, \\
&\cdots\cdots \\
|z_{m+1}(x_{n-1})-z_1(x_{n-1})| &< |z_m(x_n)-z_0(x_n)|+2^{-n}\epsilon.
\end{aligned} \tag{7}
$$

In terms of $(C3)$, (2) and (5)–(8), we deduce that

$$
\begin{aligned}
|z_{n+m}(x_0)-z_n(x_0)| &< |z_m(x_n)-z_0(x_n)|+\epsilon, \\
&\leqslant |z_m(x_n)|+|z_0(x_n)|+\epsilon \\
&\leqslant \sum_{i=0}^{m}\psi(\varphi^i(\|x_n\|))+\psi(\|x_n\|)+\epsilon \\
&\leqslant \sum_{i=0}^{m}\psi(\varphi^{i+1}(\|x_{n-1}\|))+\psi(\varphi(\|x_{n-1}\|))+\epsilon
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \sum_{i=0}^{m} \psi(\varphi^{i+n}(\|x_0\|)) + \psi(\varphi^{n}(\|x_0\|)) + \epsilon \\
&\leqslant \sum_{i=n-1}^{\infty} \psi(\varphi^{i}(\|x_0\|)) + \epsilon \\
&\leqslant \sum_{i=n-1}^{\infty} \psi(\varphi^{i}(k)) + \epsilon,
\end{aligned}$$

which yields that

$$d_k(z_{n+m}, z_n) \leqslant \sum_{i=n-1}^{\infty} \psi(\varphi^{i}(k)) + \epsilon.$$

Letting $\epsilon \to 0$ in the above inequality, we get that

$$d_k(z_{n+m}, z_n) \leqslant \sum_{i=n-1}^{\infty} \psi(\varphi^{i}(k)). \tag{8}$$

Notice that $\sum_{n=0}^{\infty} \psi(\varphi^{n}(t)) < \infty$ for each $t > 0$. Thus (8) ensures that $\{z_n\}_{n\geqslant 0}$ is a Cauchy sequence in $(BB(S), d)$ and it converges to some $z \in BB(S)$. It follows from (4) that

$$\begin{aligned}
d(Gz, z) &\leqslant d(Gz, Gz_n) + d(z_{n+1}, z) \\
&\leqslant d(z, z_n) + d(z_{n+1}, z) \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which yields that $Gz = z$. That is, the functional equation (1) possesses a solution $z$.

Given $x_0 \in S$, $\{y_n\}_{n\geqslant 1} \subset D$ and $x_n \in \{a(x_{n-1}, y_n), b(x_{n-1}, y_n), c(x_{n-1}, y_n)\}$ for $n \geqslant 1$. Put $k = [\|x_0\|] + 1$, where $[t]$ denotes the largest integer not exceeding $t$. For each $\epsilon > 0$, there exists a positive integer $m$ with

$$d_k(z, z_n) + \sum_{i=n}^{\infty} \psi(\varphi^{i}(k)) < \epsilon, \quad \forall n > m. \tag{9}$$

It follows from $(C3)$ that

$$\|x_n\| \leqslant \varphi(\|x_{n-1}\|) \leqslant \cdots \leqslant \varphi^{n}(\|x_0\|) \leqslant \varphi^{n}(k) \leqslant k, \quad \forall n \geqslant 1. \tag{10}$$

On account of (5), (9) and (10), we obtain that for $n > m$

$$
\begin{aligned}
|z(x_n)| &\leqslant |z(x_n) - z_n(x_n)| + |z_n(x_n)| \\
&\leqslant d_k(z, z_n) + \sum_{i=0}^{n} \psi(\varphi^i(\|x_n\|)) \\
&\leqslant d_k(z, z_n) + \sum_{i=0}^{n} \psi(\varphi^i(k)) \\
&< \epsilon,
\end{aligned}
$$

which gives that $\lim_{n\to\infty} z(x_n) = 0$.

Assume that $g$ is another solution of the functional equation (1) relative to condition $(C5)$. For any $\epsilon > 0$ and $x_0 \in S$, if $\mathrm{opt}_{y\in D} = \inf_{y\in D}$, then there exist $s, t \in D$ satisfying

$$
\begin{aligned}
&z(x_0) > H(x_0, s, z) - 2^{-1}\epsilon, \qquad g(x_0) > H(x_0, t, g) - 2^{-1}\epsilon, \\
&z(x_0) \leqslant H(x_0, t, z), \qquad g(x_0) \leqslant H(x_0, s, g).
\end{aligned} \tag{11}
$$

In view of Lemma 1.1, $(C1)$ and (11), we can select $y_1 \in \{s, t\} \subset D$ and $x_1 \in \{a(x_0, y_1),\ b(x_0, y_1), c(x_0, y_1)\}$ such that

$$
\begin{aligned}
&|z(x_0) - g(x_0)| \\
&\quad < \max\{|H(x_0,s,z) - H(x_0,s,g)|, |H(x_0,t,z) - H(x_0,t,g)|\} + 2^{-1}\epsilon \\
&\quad \leqslant \max\{|u(x_0,s)||\max\{p(x_0,s), z(a(x_0,s))\} - \max\{p(x_0,s), g(a(x_0,s))\}| \\
&\qquad + |v(x_0,s)||\min\{q(x_0,s), z(b(x_0,s))\} - \min\{q(x_0,s), g(b(x_0,s))\}| \\
&\qquad + |w(x_0,s)||z(c(x_0,s)) - g(c(x_0,s))|, |u(x_0,t)||\max\{p(x_0,t), z(a(x_0,t))\} \\
&\qquad - \max\{p(x_0,t), g(a(x_0,t))\}| + |v(x_0,t)||\min\{q(x_0,t), z(b(x_0,t))\} \\
&\qquad - \min\{q(x_0,t), g(b(x_0,t))\}| + |w(x_0,t)||z(c(x_0,t)) - g(c(x_0,t))|\} + 2^{-1}\epsilon \\
&\quad \leqslant \max\{(|u(x_0,s)| + |v(x_0,s)| + |w(x_0,s)|)\max\{|z(a(x_0,s)) - g(a(x_0,s))|, \\
&\qquad\qquad |z(b(x_0,s)) - g(b(x_0,s))|, |z(c(x_0,s)) - g(c(x_0,s))|\}, \\
&\qquad\qquad (|u(x_0,t)| + |v(x_0,t)| + |w(x_0,t)|)\max\{|z(a(x_0,t)) - g(a(x_0,t))|, \\
&\qquad\qquad \times |z(b(x_0,t)) - g(b(x_0,t))|, |z(c(x_0,t)) - g(c(x_0,s))|\} + 2^{-1}\epsilon \\
&\quad \leqslant |z(x_1) - g(x_1)| + 2^{-1}\epsilon,
\end{aligned}
$$

that is,

$$
|z(x_0) - g(x_0)| < |z(x_1) - g(x_1)| + 2^{-1}\epsilon; \tag{12}
$$

if $\mathrm{opt}_{y\in D} = \sup_{y\in D}$, we similarly conclude that (12) holds. Proceeding in this way, we know that there exist $y_i \in D$ and $x_i \in \{a(x_{i-1}, y_i), b(x_{i-1}, y_i), c(x_{i-1}, y_i)\}$ for $i \in \{2, 3, \ldots, n\}$ such that

$$\begin{aligned}
|z(x_1)-g(x_1)| &< |z(x_2)-g(x_2)|+2^{-2}\epsilon,\\
|z(x_2)-g(x_2)| &< |z(x_3)-g(x_3)|+2^{-3}\epsilon,\\
&\cdots\cdots\\
|z(x_{n-1})-g(x_{n-1})| &< |z(x_n)-g(x_n)|+2^{-n}\epsilon. \qquad (13)
\end{aligned}$$

It follows from $(C5)$, (12) and (13) that

$$|z(x_0)-g(x_0)|<|z(x_n)-g(x_n)|+\epsilon\to\epsilon \quad \text{as } n\to\infty.$$

Since $\epsilon$ is arbitrary, we immediately conclude that $z(x_0)=g(x_0)$. This completes the proof. □

*Remark 2.1.* (a) If $u(x,y)=v(x,y)=0$, $w(x,y)=1$ for each $(x,y)\in S\times D$, $\psi=MI$, where $M$ is a constant, and $\text{opt}_{y\in D}=\sup_{y\in D}$, then Theorem 2.1 reduces to Theorem 2.4 of Bhakta and Mitra [8].

(b) In case $v(x,y)=w(x,y)=0, u(x,y)=1$ for each $(x,y)\in S\times D$, $\psi=I$, and $\text{opt}_{y\in D}=\inf_{y\in D}$, then Theorem 2.1 reduces to Theorem 3.5 of Bhakta and Choudhury [7] and a result of Bellman [3, p. 149].

The example below reveals that Theorem 2.1 extends substantially the results due to Bellman [3], Bhakta and Choudhury [7] and Bhakta and Mitra [8].

EXAMPLE 2.1. Let $X=Y=R$ and $S=D=R^+$. Define $u,v,w,p,q,r:S\times D\to R, a,b,c:S\times D\to S$, and $\varphi,\psi:R^+\to R^+$ by

$$u(x,y)=\begin{cases}\frac{1}{2+xy}, & \text{if } x\geqslant y,\\ \frac{-1}{8+x+y}, & \text{if } x<y,\end{cases}\qquad v(x,y)=\begin{cases}-\frac{1}{4}, & \text{if } x\geqslant y,\\ \frac{1}{8+x}, & \text{if } x<y,\end{cases}$$

$$w(x,y)=\begin{cases}\frac{-1}{4+x^2+y}, & \text{if } x\geqslant y,\\ \frac{3}{5+\sin(x-y)}, & \text{if } x<y,\end{cases}\qquad p(x,y)=x^2\cos(x^2-y^2),$$

$$q(x,y)=\frac{x^3y}{1+x^2y^2},\qquad r(x,y)=\frac{x\sin x}{1+\ln(1+|x-y|)},$$

$$a(x,y)=\frac{x}{1+x^2+y^2},\qquad b(x,y)=\frac{x}{2+xy},\qquad c(x,y)=\frac{x}{3+\sin(x-y)}$$

for all $(x,y)\in S\times D$ and

$$\psi(t)=t^2 \quad\text{and}\quad \varphi(t)=\frac{1}{2}t,\quad \forall t\in R^+.$$

It is easy to verify that the assumptions of Theorem 2.1 are fulfilled. Hence the functional equation(1) possesses a solution $z\in BB(S)$ satisfying conditions

$(C4)$–$(C6)$. However, the results of Bellman[3], Bhakta and Cholldhury [7] and Bhakta and Mitra [8] are not valid for the functional equation (1).

Taking $v(x, y) = 0$ for all $(x, y) \in S \times D$ in Theorem 2.1, we have

THEOREM 2.2. *Let* $u, w, p, r : S \times D \to R$ *and* $a, c : S \times D \to S$ *be mappings and let* $(\varphi, \psi)$ *be in* $\Phi_1$ *satisfying conditions*

$(C7)$ $|u(x, y)| + |w(x, y)| \leqslant 1$ *for all* $(x, y) \in S \times D$;

$(C8)$ $\max\{|p(x, y)|, |r(x, y)|\} \leqslant \psi(\|x\|)$ *for all* $(x, y) \in S \times D$;

$(C9)$ $\max\{\|a(x, y)\|, \|c(x, y)\|\} \leqslant \varphi(\|x\|)$ *for all* $(x, y) \in S \times D$.

*Then the functional equation*

$$f(x) = \operatorname*{opt}_{y\in D}\{u(x, y)\max\{p(x, y), f(a(x, y))\} + w(x, y)[r(x, y) + f(c(x, y))]\}, \quad \forall x \in S \tag{14}$$

*possesses a solution* $z \in BB(S)$ *satisfying conditions* $(C10)$–$(C14)$

$(C10)$ *the sequence* $\{z_n\}_{n\geqslant 0}$ *defined by*

$$z_0(x) = \operatorname*{opt}_{y\in D}\{u(x, y)p(x, y) + w(x, y)r(x, y)\}, \quad \forall x \in S,$$

$$z_n(x) = \operatorname*{opt}_{y\in D}\{u(x, y)\max\{p(x, y), z_{n-1}(a(x, y))\} + w(x, y)[r(x, y) + z_{n-1}(c(x, y))]\}, \quad \forall x \in S,\ n \geqslant 1$$

*converges to* $z$;

$(C11)$ *if* $x_0 \in S$, $\{y_n\}_{n\geqslant 1} \subset D$ *and* $x_n \in \{a(x_{n-1}, y_n), c(x_{n-1}, y_n)\}$ *for each* $n \geqslant 1$, *then*

$$\lim_{n\to\infty} z(x_n) = 0,$$

$(C12)$ $z$ *is a unique solution of the functional equation* (14) *relative to condition* $(C11)$;

$(C13)$ *if* $u$ *and* $w$ *are nonnegative and* $u(x, y) + w(x, y) = 1$ *for all* $(x, y) \in S \times D$, *then for any* $\epsilon > 0$ *and* $x_0 \in S$, *there exist* $\{y_n\}_{n\geqslant 1} \subset D$ *and* $x_n \in \{a(x_{n-1}, y_n), c(x_{n-1}, y_n)\}$ *for all* $n \geqslant 1$ *such that*

$$z(x_0) \geqslant \sum_{n=0}^{\infty} w(x_n, y_{n+1})r(x_n, y_{n+1}) - \epsilon.$$

*Moreover, if* $r$ *is nonnegative, then* $z$ *is also nonnegative.*

*Proof.* It follows from Theorem 2.1 that the functional equation (14) possesses a solution $z \in BB(S)$ satisfying $(C10)$–$(C12)$. Now we show that $(C13)$ holds. Let $\epsilon$ be any positive number and $x_0 \in S$. If $\operatorname{opt}_{y\in D} = \inf_{y\in D}$, then there exist $y_1 \in D$ and $x_1 \in \{a(x_0, y_1), c(x_0, y_1)\}$ with

$$\begin{aligned}
z(x_0) &> u(x_0, y_1)\max\{p(x_0, y_1), z(a(x_0, y_1))\} \\
&\quad + w(x_0, y_1)[r(x_0, y_1) + z(c(x_0, y_1))] - 2^{-1}\epsilon \\
&\geqslant w(x_0, y_1)r(x_0, y_1) + u(x_0, y_1)z(a(x_0, y_1)) \\
&\quad + w(x_0, y_1)z(c(x_0, y_1)) - 2^{-1}\epsilon \\
&\geqslant w(x_0, y_1)r(x_0, y_1) + [u(x_0, y_1) + w(x_0, y_1)] \\
&\quad \times \min\{z(a(x_0, y_1)), z(c(x_0, y_1))\} - 2^{-1}\epsilon \\
&= w(x_0, y_1)r(x_0, y_1) + z(x_1) - 2^{-1}\epsilon. \qquad (15)
\end{aligned}$$

Similarly we can select $y_i \in D$ and $x_i \in \{a(x_{i-1}, y_i), c(x_{i-1}, y_i)\}$ for $i \in \{2, 3, \ldots, n\}$ such that

$$\begin{aligned}
z(x_1) &> w(x_1, y_2)r(x_1, y_2) + z(x_2) - 2^{-2}\epsilon, \\
z(x_2) &> w(x_2, y_3)r(x_2, y_3) + z(x_3) - 2^{-3}\epsilon, \\
&\cdots\cdots \\
z(x_{n-1}) &> w(x_{n-1}, y_n)r(x_{n-1}, y_n) + z(x_n) - 2^{-n}\epsilon. \qquad (16)
\end{aligned}$$

Combining (15) and (16) we derive that

$$z(x_0) > \sum_{i=1}^{n} w(x_{i-1}, y_i)r(x_{i-1}, y_i) + z(x_n) - \epsilon. \qquad (17)$$

Using the same argument as above, we also conclude that (17) holds for $\text{opt}_{y\in D} = \sup_{y\in D}$. Note that $(C8)$ and $(C9)$ ensure that

$$\begin{aligned}
|w(x_{n-1}, y_n)r(x_{n-1}, y_n)| &= w(x_{n-1}, y_n)|r(x_{n-1}, y_n)| \\
&\leqslant \psi(\|x_{n-1}\|) \leqslant \psi(\varphi(\|x_{n-2}\|)) \leqslant \cdots \\
&\leqslant \psi(\varphi^{n-1}(\|x_0\|))
\end{aligned}$$

and $\sum_{n=1}^{\infty} \psi(\varphi^{n-1}(\|x_0\|))$ is convergent. It follows that the series

$$\sum_{n=1}^{\infty} |w(x_{n-1}, y_n)r(x_{n-1}, y_n)|$$

is convergent. Letting $n \to \infty$ in (17), by $(C11)$ we know that

$$z(x_0) \geqslant \sum_{n=1}^{\infty} w(x_{n-1}, y_n)r(x_{n-1}, y_n) - \epsilon.$$

Suppose that $r$ is nonnegative. It follows from (17) that for given $\epsilon > 0$ and $x_0 \in S$, there exist $y_i \in D$ and $x_i \in \{a(x_{i-1}, y_i), c(x_{i-1}, y_i)\}$ for $i \in \{1, 2, \ldots, n\}$ such that

$$z(x_0) > z(x_n) - \epsilon. \tag{18}$$

Letting $n \to \infty$ in (18), by $(C11)$ we obtain that

$$z(x_0) \geqslant -\epsilon,$$

which gives that $z(x_0) \geqslant 0$ since $\epsilon$ is arbitrary. This completes the proof. □

A proof similar to that of Theorem 2.2 yields the following result and is thus omitted.

THEOREM 2.3. *Let* $u, w, p, r : S \times D \to R$ *and* $a, c : S \times D \to S$ *be mappings and* $(\varphi, \psi)$ *be in* $\Phi_1$ *satisfying conditions* $(C7)$–$(C9)$. *Then the functional equation*

$$\begin{aligned} f(x) = opt_{y\in D}\{&u(x, y)\min\{p(x, y), f(a(x, y))\} \\ &+ w(x, y)[r(x, y) + f(c(x, y))]\}, \quad \forall x \in S \end{aligned} \tag{19}$$

*possesses a solution* $z \in BB(S)$ *satisfying conditions* $(C11)$ *and*

$(C14)$ *the sequence* $\{z_n\}_{n\geqslant 0}$ *defined by*

$$\begin{aligned} z_0(x) &= \operatorname*{opt}_{y\in D}\{u(x, y)p(x, y) + w(x, y)r(x, y)\}, \quad \forall x \in S, \\ z_n(x) &= \operatorname*{opt}_{y\in D}\{u(x, y)\min\{p(x, y), z_{n-1}(a(x, y))\} \\ &\quad + w(x, y)[r(x, y) + z_{n-1}(c(x, y))]\}, \quad \forall x \in S,\ n \geqslant 1 \end{aligned}$$

*converges to* $z$;

$(C15)$ $z$ *is a unique solution of the functional equation* (19) *relative to condition* $(C11)$;

$(C16)$ *if* $u$ *and* $w$ *are nonnegative, then for any* $\epsilon > 0$ *and* $x_0 \in S$, *there exist* $\{z_n\}_{n\geqslant 1} \subset D$ *and* $x_n \in \{a(x_{n-1}, y_n), c(x_{n-1}, y_n)\}$ *for all* $n \geqslant 1$ *such that*

$$z(x_0) \leqslant \sum_{n=0}^{\infty} w(x_n, y_{n+1})r(x_n, y_{n+1}) + \epsilon.$$

*Moreover, if* $r$ *is nonpositive, then* $z$ *is also nonpositive.*

*Remark 2.2.* (a) Taking $\psi = I$, $u(x, y) = \lambda$, $w(x, y) = 1 - \lambda$ and $a(x, y) = c(x, y)$ for any $(x, y) \in S \times D$, where $\lambda$ is a constant in [0, 1], then Theorem 2.2 reduces to a result which improves Theorem 3.2 of Liu and Ume [16], which, in turn, is a generalization of a result of Bellman [3, p. 149], Theorem 3.5 of Bhakta and Choudhury [7] and Theorem 2.4 of Bhakta and Mitra [8].

(b) If $\psi = I$, $u(x,y)=\lambda$, $w(x,y)=1-\lambda$ and $a(x,y)=c(x,y)$ for any $(x,y)\in S\times D$, where $\lambda$ is a constant in $[0,1]$, then Theorem 2.3 reduces to a result which improves Theorem 3.3 of Liu and Ume [16].

The following examples show that Theorems 2.2 and 2.3 are indeed extensions of the results due to Bellman [3], Bhakta and Choudhury [7], Bhakta and Mitra [8] and Liu and Ume [16].

EXAMPLE 2.2. Let $X=Y=S=R$ and $D=R^+$. Define $u, w, p, r : S\times D\to R$, $a, c : S\times D\to S$ and $\varphi, \psi : R^+\to R^+$ by

$$p(x,y)=x^2\sin x, \qquad r(x,y)=\frac{2|x|^3 y}{1+y^2},$$
$$a(x,y)=\frac{x}{3+|x|y}, \qquad c(x,y)=\begin{cases}\frac{x}{3}, & \text{if } |x|<3,\\ \frac{\sin x}{|x|+y}, & \text{if } |x|\geqslant 3,\end{cases}$$
$$u(x,y)=\begin{cases}\frac{|xy|}{x^2+y^2}, & \text{for } x^2+y^2>1,\\ \frac{1-x^2y^2}{1+x^2y^2}, & \text{for } x^2+y^2\leqslant 1,\end{cases}$$
$$w(x,y)=\begin{cases}\frac{x^2+y^2-|xy|}{x^2+y^2}, & \text{for } x^2+y^2>1,\\ \frac{x^2y^2}{1+x^2y^2}, & \text{for } x^2+y^2\leqslant 1\end{cases}$$

for all $(x,y)\in S\times D$,

$$\psi(t)=t^3 \quad\text{and}\quad \varphi(t)=\frac{1}{3}t, \quad \forall t\in R^+.$$

Then the conditions of Theorem 2.2 are fulfilled and consequently the functional equation (14) possesses a solution $z\in BB(S)$ satisfying conditions $(C10)$–$(C13)$. But the results of Bellman [3], Bhakta and Choudhury [7], Bhakta and Mitra [8] and Liu and Ume [16] are not valid for the functional equation (14).

EXAMPLE 2.3. Let $X, Y, S, T, u, w, p, a, c, \varphi$ and $\psi$ be as in Example 2.2. Define $r : S\times D\to R$ by

$$r(x,y)=x^2\left|\sin\left(x+\frac{1}{1+|x|y}\right)\right|, \quad \forall (x,y)\in S\times D.$$

It is easily verified that the assumptions of Theorem 2.3 are satisfied. Thus the functional equation (19) possesses a solution in $BB(S)$ satisfying conditions $(C14)$–$(C16)$. However, the result of Liu and Ume [16] is not applicable because

$$|r(x,y)|\leqslant M|x|$$

does not hold for $(x_M, y_M) = (M+1, 1) \in S \times D$, where $M$ is a positive constant.

As consequences of Theorems 2.1–2.3, we have the following

COROLLARY 2.1. *Let* $u, p : S \times D \to R$ *and* $a : S \times D \to S$ *be mappings and let* $(\varphi, \psi)$ *be in* $\Phi_1$ *satisfying the following conditions:*
$(C17)$ $|u(x, y)| \leqslant 1$ *for all* $(x, y) \in S \times D$;
$(C18)$ $|p(x, y)| \leqslant \psi(\|x\|)$ *for all* $(x, y) \in S \times D$;
$(C19)$ $\|a(x, y)\| \leqslant \varphi(\|x\|)$ *for all* $(x, y) \in S \times D$.
*Then the functional equation*

$$f(x) = \operatorname*{opt}_{y \in D}\{u(x, y)[p(x, y) + f(a(x, y))], \quad \forall x \in S \tag{20}$$

*possesses a solution* $z \in BB(S)$ *satisfying*
$(C20)$ *the sequence* $\{z_n\}_{n \geqslant 0}$ *defined by*

$$z_0(x) = \operatorname*{opt}_{y \in D}\{u(x, y)p(x, y)\}, \quad \forall x \in S,$$
$$z_n(x) = \operatorname*{opt}_{y \in D}\{u(x, y)[p(x, y) + z_{n-1}(a(x, y))]\}, \quad \forall x \in S,\ n \geqslant 1$$

*converges to* $z$;
$(C21)$ *if* $x_0 \in S$, $\{y_n\}_{n \geqslant 1} \subset D$ *and* $x_n = a(x_{n-1}, y_n)$ *for each* $n \geqslant 1$, *then*

$$\lim_{n \to \infty} z(x_n) = 0;$$

$(C22)$ $z$ *is a unique solution of the functional equation* (20) *relative to* $(C21)$;
$(C23)$ *if* $u(x, y) = 1$ *for each* $(x, y) \in S \times D$, *then for any* $\epsilon > 0$ *and* $x_0 \in S$, *there exist* $\{y_n\}_{n \geqslant 1} \subset D$ *and* $x_n = a(x_{n-1}, y_n)$ *for all* $n \geqslant 1$ *such that*

$$\left| z(x_0) - \sum_{n=0}^{\infty} p(x_n, y_{n+1}) \right| \leqslant \epsilon.$$

*Furthermore, if p is nonnegative, then z is nonnegative; and if p is nonpositive, then z is nonpositive.*

*Proof.* In order to show Corollary 2.1, by Theorems 2.2 and 2.3 it is sufficient to show that $(C23)$ holds. Given $\epsilon > 0$ and $x_0 \in S$. If $\operatorname{opt}_{y \in D} = \inf_{y \in D}$, then there exist $\{y_n\}_{n \geqslant 1} \subset D$ and $x_n = a(x_{n-1}, y_n)$ for all $n \geqslant 1$ such that

$$p(x_{n-1}, y_n) + z(x_n) - 2^{-n}\epsilon < z(x_{n-1}) \leqslant p(x_{n-}, y_n) + z(x_n),$$

which yields that

$$\sum_{i=0}^{n-1} p(x_i, y_{i+1}) + z(x_n) - \epsilon < z(x_0) \leqslant \sum_{i=0}^{n-1} p(x_i, y_{i+1}) + z(x_n). \tag{21}$$

Letting $n \to \infty$ in (21), by ($C21$) we know that

$$\left| z(x_0) - \sum_{n=0}^{\infty} p(x_n, y_{n+1}) \right| \leqslant \epsilon.$$

Similarly we conclude that the above inequality holds for $\mathrm{opt}_{y\in D} = \sup_{y\in D}$. This completes the proof. □

The proofs of Corollaries 2.2 and 2.3 are quite similar to those of Theorems 2.1 and 2.2 and Corollary 2.1, and therefore are omitted.

COROLLARY 2.2. *Let $u, p: S \times D \to R$ and $a: S \times D \to S$ be mappings and let $(\varphi, \psi)$ be in $\Phi_2$ satisfying conditions* ($C17$)–($C19$). *Then the functional equation*

$$f(x) = \mathop{\mathrm{opt}}_{y\in D}\{u(x, y) \max\{p(x, y), f(a(x, y))\}\}, \quad \forall x \in S \tag{22}$$

*possesses a solution $z \in BB(S)$ satisfying conditions* ($C21$) *and*

($C24$) *the sequence $\{z_n\}_{n\geqslant 0}$ defined by*

$$z_0(x) = \mathop{\mathrm{opt}}_{y\in D}\{u(x, y)p(x, y)\}, \quad \forall x \in S,$$
$$z_n(x) = \mathop{\mathrm{opt}}_{y\in D}\{u(x, y) \max\{p(x, y), z_{n-1}(a(x, y))\}\}, \quad \forall x \in S,\ n \geqslant 1$$

*converges to z;*

($C25$) *z is a unique solution of the functional equation* (22) *relative to condition* ($C21$);

($C26$) *if $u(x, y) = 1$ for each $(x, y) \in S \times D$, then z is nonnegative.*

COROLLARY 2.3. *Let $u, p: S \times D \to R$ and $a: S \times D \to S$ be mappings and let $(\varphi, \psi)$ be in $\Phi_2$ satisfying conditions* ($C17$)–($C19$). *Then the functional equation*

$$f(x) = \mathop{\mathrm{opt}}_{y\in D}\{u(x, y) \min\{p(x, y), f(a(x, y))\}\}, \quad \forall x \in S \tag{23}$$

*possesses a solution $z \in BB(S)$ satisfying condition* ($C21$) *and*

($C27$) *the sequence* $\{z_n\}_{n\geqslant 0}$ *defined by*

$$z_0(x)=\operatorname*{opt}_{y\in D}\{u(x,y)p(x,y)\},\quad \forall x\in S,$$

$$z_n(x)=\operatorname*{opt}_{y\in D}\{u(x,y)\min\{p(x,y),z_{n-1}(a(x,y))\}\},\quad \forall x\in S,\ n\geqslant 1$$

*converges to* $z$;

($C28$) $z$ *is a unique solution of the functional equation* (23) *relative to condition* ($C21$);

($C29$) *if* $u(x,y)=1$ *for each* $(x,y)\in S\times D$, *then* $z$ *is nonpositive.*

*Remark 2.3.* In case $u(x,y)=1$ for all $(x,y)\in S\times D$, $\psi=MI$, where $M$ is a positive constant, then Corollaries 2.1-2.3 reduce to three results which include Theorem 2.4 of Bkakta and Choudhury [7], Theorem 3.5 of Bhakta and Mitra [8], Theorem 3.5 of Liu [14] and Corollaries 3.2–3.4 of Liu and Ume [16] as special cases.

The following simple example demonstrates that Corollaries 2.1–2.3 generalize properly the results due to Bhakta and Choudhury [7], Bhakta and Mitra [8], Liu [14] and Liu and Ume [16].

EXAMPLE 2.4. Let $X=Y=R$, $S=R^+$ and $D=R^-$. Define $u:S\times D\to R$ and $\psi,\varphi:R^+\to R^+$ by

$$u(x,y)=1,\ \forall(x,y)\in S\times D,\ \psi(t)=t^2 \text{ and } \varphi(t)=\frac{t}{2},\ \forall t\in R^+.$$

According to Corollaries 2.1–2.3, we immediately conclude that the functional equations

$$f(x)=\operatorname*{opt}_{y\in D}\left\{x^2\sin y+f\left(\frac{x}{2-xy}\right)\right\},\quad \forall x\in S, \tag{24}$$

$$f(x)=\operatorname*{opt}_{y\in D}\max\left\{x^2\sin y,\quad f\left(\frac{x}{2-xy}\right)\right\},\quad \forall x\in S \tag{25}$$

and

$$f(x)=\operatorname*{opt}_{y\in D}\min\left\{x^2\sin y,\quad f\left(\frac{x}{2-xy}\right)\right\},\quad \forall x\in S \tag{26}$$

possess, respectively, a solution in $BB(S)$ satisfying conditions ($C20$)–($C23$), conditions ($C21$) and ($C24$)–($C26$), and conditions ($C21$) and ($C27$)–($C29$), respectively. But the results of Bhakta and Choudhury [7], Bhakta and Mitra [8], Liu [14] and Liu and Ume [16] are not valid for the functional equations (24)–(26).

$f^*$ and $g^*$ are called coincidence solutions of the following system of functional equations

$$\begin{cases} f(x) = \text{opt}_{y\in D}\{u(x,y)\max\{p(x,y), g(a(x,y))\} \\ \qquad +v(x,y)\min\{q(x,y), g(b(x,y))\} \\ \qquad +w(x,y)[r(x,y)+g(c(x,y))]\}, \\ g(x) = \text{opt}_{y\in D}\{u(x,y)\max\{p(x,y), f(a(x,y))\} \\ \qquad +v(x,y)\min\{q(x,y), f(b(x,y))\} \\ \qquad +w(x,y)[r(x,y)+f(c(x,y))]\} \end{cases}, \quad \forall x\in S \tag{27}$$

if condition (28) below is fulfilled :

$$\begin{aligned} f^*(x) &= \text{opt}_{y\in D}\{u(x,y)\max\{p(x,y), g^*(a(x,y))\} \\ &\quad +v(x,y)\min\{q(x,y), g^*(b(x,y))\} \\ &\quad +w(x,y)[r(x,y)+g^*(c(x,y))]\}, \\ g^*(x) &= \text{opt}_{y\in D}\{u(x,y)\max\{p(x,y), f^*(a(x,y))\} \\ &\quad +v(x,y)\min\{q(x,y), f^*(b(x,y))\} \\ &\quad +w(x,y)[r(x,y)+f^*(c(x,y))]\} \end{aligned}, \quad \forall x\in S. \tag{28}$$

Before concluding this paper, we would like to point out directions for further work in the form of open problems.

**Problem 2.1.** Is there a contraction mapping $G: BB(S)\to BB(S)$ such that the Picard iteration sequence $\{z_n\}_{n\geqslant 0}$ defined by (C4) converges to the unique fixed point of $G$ under the conditions of Theorem 2.1 ?

**Problem 2.2.** Is there a nonnegative or nonpositive solution for the functional equation (1) under the conditions of Theorem 2.1 ?

**Problem 2.3.** If the answer to Problem 2.2 is no, then what additional hypotheses are needed in Theorem 2.1 to guarantee the existence of nonnegative or nonpositive solutions for the functional equation (1) ?

**Problem 2.4.** Does the system of functional equations (27) possess coincidence solutions in $BB(S)$ ?

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